

Analytic Perturbation Theory for QCD observables

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Abstract

The connection between ghost-free formulations of RG-invariant perturbation theory in the both Euclidean and Minkowskian regions is investigated. Our basic tool is the “double spectral representation”, similar to the definition of Adler function, that stems from first principles of local QFT. It relates real functions defined in the Euclidean and Minkowskian regions.

On this base we establish a simple relation between

— The trick of resummation of the π^2 -terms (known from early 80s) for the invariant QCD coupling and observables in the time-like region and

— Invariant Analytic Approach (devised a few years ago) with the “analyticized” coupling $\alpha_{\text{an}}(Q^2)$ and nonpower perturbative expansion for observables in the space-like domain which are free of unphysical singularities .

As a result, we formulate a self-consistent scheme — Analytic Perturbation Theory (APT)

— that relates a renorm-invariant, effective coupling functions $\alpha_{\text{an}}(Q^2)$ and $\tilde{\alpha}(s)$, as well as non-power perturbation expansions for observables in both space- and time-like domains, that are free of extra singularities and obey better convergence in the infrared region.

Then we consider the issue of the heavy quark thresholds and devise a global APT scheme for the data analysis in the whole accessible space-like and time-like domain with various numbers of active quarks.

Preliminary estimates indicate that this global scheme produces results a bit different, sometimes even in the five-flavour region, on a few per cent level for $\bar{\alpha}_s$ – from the usual one, thus influencing the total picture of the QCD parameter correlation.

1 Introduction

1.1 Preamble: perturbation theory and $\bar{\alpha}_s$

The issue of the strong interaction behavior at the low and medium energy $W = \sqrt{s}$ and momentum transfer $Q = \sqrt{Q^2}$ attracts more and more interest along with the further experimental data accumulation.

As a dominant means of theoretical analysis, here one uses the perturbative QCD (pQCD), in spite of the fact that in the given domain the power expansion parameter $\bar{\alpha}_s$ is not a “small enough” quantity. Physically, this region corresponds to three ($f = 3$) and four ($f = 4$) flavors (active quarks). Just in the three-flavour region there lie unphysical singularities of central theoretical object — invariant effective coupling $\bar{\alpha}_s$. These singularities, associated with the scale parameter $\Lambda_{f=3} \simeq 350$ MeV, complicate theoretical interpretation of data in the “small energy” and “small momentum transfer” regions ($\sqrt{s}, Q \equiv \sqrt{Q^2} \lesssim 3\Lambda_3$.) On the other hand, their existence contradicts some general statements of the local QFT.

It is important to notice that in the current literature for the effective QCD coupling in the time-like domain $\alpha_s(s)$; $s = W^2$ one uses literally the same expression, like one in the Euclidean domain. By the way, implanting the mentioned singularities into the three-flavor region of small energies $W \simeq 350$ MeV.

Meanwhile, the notion of invariant electron charge (squared) $\bar{\alpha}(Q^2) = \bar{e}^2(Q)$ in QED has initially been defined in the early papers[1] on renormalization group (RG) only in the space-like region in terms of a product of real constants z_i of finite Dyson renormalization transformation. Just the Euclidean invariant charge $\bar{e}(Q)$ is related by the Fourier transformation to the space distribution $\bar{e}(r)$ of the electric charge (around a point “bare” electron) introduced by Dirac[2].

Analogous motivation in the RG formalism (for detail, see chapter “Renormalization group” in the text[3]) underlies a more general notion of invariant coupling $\bar{g}(Q)$, *defined only in the space-like domain*. Inside the RG formalism, there is no simple means for defining \bar{g} in the time-like region.

Nevertheless, in modern practice, inspired by “highly authoritative reviews”[4, 5] and some monographs (like [6]) one uses the same singular expression for the QCD effective coupling $\bar{\alpha}_s$ both in the space- and time-like domains.

Technically, this “implanting” of the $\bar{\alpha}_s$ Euclidean functional form into Minkowskian is accompanied by some modification of numerical expansion coefficients. To the initial coefficient calculated by Feynman diagrams, one adds specific terms (containing π^2 and its powers) with coefficients of some lower orders. These “ π^2 -terms” are the only “atonement for the Styx river crossing” from the Euclid realm to the Minkowski domain.

1.2 Time-like region, π^2 -terms

Meanwhile, as it easy to show, the “ π^2 -procedure” is valid only at small parameter $\pi^2/\ln^2(s/\Lambda^2)$ values, that is in the region of high enough energies $W \gg \Lambda e^{\pi/2} \simeq 3$ GeV.

Here, it is useful to restore the ideas proposed by Radyushkin[7] and Krasnikov—Pivovarov[8] (RKP procedure) at the beginning of the 80s.

To introduce an invariant $\bar{\alpha}_s$ in the time-like region, both the authors used integral transformation \mathbf{R} , “reverse” to the Adler function definition. The last one can be treated as the definition of integral operation

$$R(s) \rightarrow D(z) = Q^2 \int_0^\infty \frac{ds}{(s+z)^2} R(s) \equiv \mathbf{D} \{R(s)\} , \quad (1)$$

transforming a real function $R(s)$ of positive (time-like) argument into the function $D(z)$ defined in the cut complex plane with analytic properties adequate to the Källén–Lehmann representation. In particular, $D(Q^2)$ is real at the positive semiaxis $z = Q^2 + i0$; $Q^2 \geq 0$.

The reverse operation \mathbf{R} can be expressed via the contour integral

$$R(s) = \frac{i}{2\pi} \int_{s-i\varepsilon}^{s+i\varepsilon} \frac{dz}{z} D_{\text{pt}}(-z) \equiv \mathbf{R} [D(Q^2)] .$$

By operation \mathbf{R} one can define RG-invariant, effective coupling $\tilde{\alpha}(s) = \mathbf{R} [\bar{\alpha}_s(Q^2)]$ in the time-like region. A few simple examples are in order

— For the one-loop effective coupling $\bar{\alpha}_s^{(1)} = [\beta_0 \ln(Q^2/\Lambda^2)]^{-1}$ one has [9, 7, 10]¹

$$\mathbf{R} [\bar{\alpha}_s^{(1)}] = \tilde{\alpha}^{(1)}(s) = \frac{1}{\beta_0} \left[\frac{1}{2} - \frac{1}{\pi} \arctan \frac{L}{\pi} \right]_{L>0} = \frac{1}{\beta_0 \pi} \arctan \frac{\pi}{L}; \quad L = \ln \frac{s}{\Lambda^2} . \quad (2)$$

— At the two-loop case, to the popular approximation

$$\beta_0 \bar{\alpha}_{s, \text{pop}}^{(2)}(Q^2) = \frac{1}{l} - b_1(f) \frac{\ln l}{l^2}; \quad l = \ln \frac{Q^2}{\Lambda^2}$$

there corresponds [7, 11]

$$\tilde{\alpha}_{\text{pop}}^{(2)}(s) = \left(1 + \frac{b_1 L}{L^2 + \pi^2} \right) \tilde{\alpha}^{(1)}(s) - \frac{b_1 \ln [\sqrt{L^2 + \pi^2}] + 1}{\beta_0 (L^2 + \pi^2)} . \quad (3)$$

Both the expressions (2) and (3) are monotonously decreasing with finite IR limit $\tilde{\alpha}(0) = 1/\beta_0(f=3) \simeq 1.4$.

— At the same time, square and cube of $\bar{\alpha}_s^{(1)}$ transform into simple “pipized” expressions[7, 8]

$$\mathfrak{A}_2^{(1)}(s) \equiv \mathbf{R} [(\bar{\alpha}_s^{(1)})^2] = \frac{1}{\beta_0^2 [L^2 + \pi^2]} \quad \mathfrak{A}_3^{(1)}(s) = \frac{L}{\beta_0^3 [L^2 + \pi^2]^2} , \quad (4)$$

which *are not powers* of $\tilde{\alpha}^{(1)}(s)$.

¹Later on, this idea has been discussed by several known authors — see Refs. [12] — [15].

Note also that transition from singular $\bar{\alpha}_s$ and its powers to “pipized” expressions, that is operation \mathbf{R} , can be performed [12, 14] by the differential operator

$$\mathcal{R} = \frac{\sin \pi \mathcal{P}}{\pi \mathcal{P}}; \quad \mathcal{P} = Q^2 \frac{d}{dQ^2} \quad \text{with the substitution} \quad Q^2 \rightarrow s, \quad \text{e.g.,} \quad \mathcal{R}\bar{\alpha}_s(Q^2) = \tilde{\alpha}(s).$$

The most remarkable feature of all presented expressions for $\tilde{\alpha}(s)$ and $\mathfrak{A}_k(s)$ (valid in a more general case) is the absence of unphysical singularity (the “log pole” at the one-loop case)² which is “screened” by π^2 -contributions.

Besides, a common “Euclidean” perturbation expansion

$$D_{\text{pt}}(Q^2) = 1 + \sum_{k \geq 1} d_k \bar{\alpha}_s^k(Q^2) \quad (5)$$

in powers of the standard RG-summed effective coupling $\bar{\alpha}_s(Q^2)$, with its unphysical singularities in the IR region (at $Q^2 \leq \Lambda_3^2$) being transformed by \mathbf{R} to the time-like region, transits into the asymptotic expansion over a nonpower set of functions

$$R_\pi(s) \equiv \mathbf{R} [D_{\text{pt}}(Q^2)] = 1 + \sum_{k \geq 1} d_k \mathfrak{A}_k(s); \quad \mathfrak{A}_k(s) = \mathbf{R} [\bar{\alpha}_s^k(Q^2)] , \quad (6)$$

with better properties of decreasing[7] of subsequent terms .

At the same time, higher functions, like (4), vanish $\mathfrak{A}_k(0) = 0$; $k \geq 1$ in the IR limit.

On the other hand, in the UV region at $\ln(s/\Lambda^2) \gg \pi$, i.e., for $W \gg \Lambda e^{\pi/2} \simeq 3 \text{ GeV}$, the functions $\tilde{\alpha}$ and \mathfrak{A}_k can be represented as a series in powers of the parameter π^2/L^2 , $L = \ln(s/\Lambda^2)$. Such expressions sometimes can be reformulated into expansions in powers of α_s . For instance, in the one-loop case

$$\tilde{\alpha}^{(1)}(s) \simeq \frac{1}{\beta_0 L} - \frac{\pi^2}{3L^2} + \frac{\pi^4}{5L^4} + \simeq \bar{\alpha}_s^{(1)}(s) - \frac{\pi^2 \beta_0^2}{3} (\bar{\alpha}_s^{(1)}(s))^3 + \frac{\pi^4 \beta_0^4}{5} (\bar{\alpha}_s^{(1)}(s))^5 \quad (7)$$

Without going into detail, note³ that, qualitatively, the functions \mathfrak{A}_k behave very similarly to the functions \mathcal{A}_k involved into non-power Euclidean asymptotic expansion for observables[18] arising in the Analytic Perturbation Theory (APT) – see below Section 1.3 and Figure 2. In particular, they oscillate at small argument values and form an asymptotic set à la Erdélyi.

As it follows from eqs. (2) and (4), one-loop “pipized” functions satisfy the recursion relation $(d/dL)\mathfrak{A}_k^{(1)}(s) = -k \beta_0 \mathfrak{A}_{k+1}^{(1)}(s)$ which is analogous to the one-loop differential equation for the invariant coupling. According to [18], this recursion is valid for analyticized functions $\mathcal{A}_k^{(1)}$.

²This feature was not mentioned in the pioneer papers of the 80s we have cited above.

³See, also the first version of this paper [16]. A more minute numerical information on functions $\tilde{\alpha}$, α_{an} , $\mathfrak{A}_{2,3}$ and $\mathcal{A}_{2,3}$ can be found in recent paper by Magradze [17].

Quite recently [17] the two-loop generalization has been found

$$\frac{1}{k} \frac{d}{dL} \mathfrak{A}_k^{(2)}(s) = -\beta_0 \mathfrak{A}_{k+1}^{(2)}(s) - \beta_1 \mathfrak{A}_{k+2}^{(2)}(s). \quad (8)$$

It can be considered as (at $k = 1$) a mould of two-loop differential equation for $\bar{\alpha}_s$. Analogous relation is valid for analyticized \mathcal{A}_k .

For the reverse transition from Minkowski to Euclid, one could try to use the transformation **D** defined by (1). However, it is evident that we shall not return to the initial coupling $\bar{\alpha}_s$ and to series in its powers (5). To elucidate the issue, it is useful to turn to the foundation of the Invariant Analytic Approach mentioned above.

1.3 Space-like region : APT

Indeed, as it has been well-known from the late 50s [19], there exists a method of getting rid of Euclidean unphysical singularities by combining RG-summed expressions with Källén–Lehmann analytical representation for $\bar{\alpha}_s(Q^2)$ in the Q^2 variable. In the mid-90s this idea was used in QCD [20, 21, 22] under the name of Invariant Analytic Approach. Its further development and application to perturbative expansion for observables yielded Analytic Perturbation Theory — [23].

We remind here the basic features and results of APT (— see also a recent review [24]).

By combining three elements

1. Usual Feynman perturbation theory for effective coupling(s) and observables,
2. Renormalizability, i.e., renormalization-group (RG) invariance, and
3. General principles of local QFT — like causality, unitarity, Poincaré invariance and spectrality — in the form of spectral representations of the Källén–Lehmann and Jost–Lehmann–Dyson type

it turns out to be possible to formulate an *Invariant Analytic Approach* (IAA) for the pQCD invariant coupling and observables in which the central theoretical object is a *spectral density*.

- Being calculated by the usual RG-improved perturbation theory, it defines and relates Q^2 -analytic, RG-invariant expressions for effective RG-invariant coupling and perturbative observables in the Euclidean channel.
- In particular, the IAA results in the modified ghost-free expression for the invariant QCD coupling $\alpha_{\text{an}}(Q^2; f)$ which is free of ghost troubles and obey reduced [21] – [28] higher-loops and renormalization-scheme sensitivity⁴. See, Fig.1.

⁴This analyticized QCD coupling α_{an} has been successively used [29, 30] in analysis of the pion and $\gamma^* \gamma \rightarrow \pi^0$ formfactors.

- The IAA change the structure of perturbation expansion for observables: Instead of common power series, as a result of integral transformation, there appears non-power asymptotic expansion [18] á la Erdélyi over the sets of specific functions $\mathcal{A}_k(Q^2; f)$, free of unphysical ghosts. These functions are defined via integral transformations of related powers $\alpha_s^k(Q^2; f)$ in terms of relevant spectral densities. This nonpower expansion for an observable, with the coefficients extracted from the relevant Feynman diagrams, we call the **Analytic Perturbation Theory**.

At small and moderate arguments, \mathcal{A}_k diminish with the k growth much quicker than the powers of α_{an}^k (and even oscillate in the region $\sqrt{s}, Q \simeq \Lambda$), thus improving essentially the convergence of perturbation expansion for observables.

The first purpose of this work is to elucidate relation between the Radyushkin–Krasnikov–Pivovarov procedure leading to effective summation of π^2 -terms (“pipization” trick)[7, 8] for observables and the Solovtsov[10, 15] construction of the effective QCD coupling within the IAA scheme in the s -channel.

In the course of this analysis — see Section 2 — we discuss the APT proliferation to the time-like region, remind a spectacular effect of “distorting mirror” correlation[27] between analyticized and pipized invariant QCD couplings in space-like $\alpha_{\text{an}}(Q^2; f)$ and time-like $\tilde{\alpha}(s; f)$ regions (see Fig.1 below), and establish this effect for corresponding expansion functions $\mathcal{A}_k(Q^2; f)$ and $\mathfrak{A}_k(s; f)$ — see Fig.2.

Then, in Section 3, we consider an the transition across the heavy quark thresholds, to construct a “global” picture for the whole physical region $M_\tau \lesssim \sqrt{s}, Q \lesssim M_Z$ — see Fig.2.

It should be noted, that all precedent papers Refs.[8] – [36] dealt only with the massless quarks in the case with fixed flavour number f . This can be justified, to some extent, when analyzing inside a narrow interval of the relevant energy \sqrt{s} or momentum transfer Q values. Meanwhile, the ultimate goal of all the pQCD is a correlation of effective coupling values extracted from different experiments.

Main results of this investigation are reviewed in the Conclusion.

2 Self-consistent scheme for observables

2.1 Modification of the APT

As it has been mentioned above, applying operation **D** to $\tilde{\alpha}$ does not restore a usual effective coupling as far as representation (1) is not compatible with ghost singularity of $\bar{\alpha}_s$.

Instead, we arrive at

$$\mathbf{D} \{ \tilde{\alpha}(s; f) \} = \frac{Q^2}{\pi} \int_0^\infty \frac{ds}{(s + Q^2)^2} \tilde{\alpha}(s; f) \equiv \alpha_{\text{an}}(Q^2; f) , \quad (9)$$

i.e., to effective Euclidean coupling α_{an} of APT. This simple fact has first been established in [10]. We see that operations \mathbf{D} and \mathbf{R} relate “pipized” and “analyticized” coupling functions in space- and time-like regions. Hence, in this case $\mathbf{R} = \mathbf{D}^{-1}$. Note, however, that the relation $\mathbf{DR} = \mathbf{1}$ is valid *only for the class of functions* $F(Q^2) \in \mathbf{C}_{KL}$ *satisfying the Källen–Lehmann representation*.

Now, we have the possibility of extending the APT to the time-like region. We shall do it in the form of a recipe, using operation of analyticization

$$F(Q^2) \rightarrow F_{\text{an}}(Q^2) = \mathbf{A} \cdot F(Q^2), \quad (10)$$

first introduced in Refs.[20, 21] in terms of the Källen–Lehmann representation

$$F_{\text{an}}(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma + Q^2} \rho_{\text{pt}}(\sigma); \quad \rho_{\text{pt}}(\sigma) = \Im F(-\sigma), \quad (11)$$

with spectral density defined via straightforward continuation of F on the cut.

Relations (10) and (11) together define the analyticization operation. Now, we can formulate the APT anew.

Firstly, one has to transform the common singular coupling function $\bar{\alpha}_s(Q^2)$ or some power expansion of an observable

$$D_{\text{pt}}(Q^2) = 1 + \sum_{k \geq 1} d_k \bar{\alpha}_s^k(Q^2; f) \quad (12)$$

into the corresponding analytic Euclidean expression α_{an} or $D_{\text{an}}(Q^2)$, free of ghosts

$$D_{\text{an}}(Q^2; f) = 1 + \sum_{k \geq 1} d_k \mathcal{A}_k(Q^2; f); \quad \alpha_{\text{an}}(Q^2; f) = \mathcal{A}_1(Q^2; f), \quad (13)$$

$$\mathcal{A}_k(x; f) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma + x} \rho_k(\sigma; f); \quad \rho_k(\sigma; f) = \Im [\bar{\alpha}_s^k(-\sigma; f)] \quad (14)$$

with spectral densities ρ, ρ_k introduced according to (11).

Secondly, by operation \mathbf{R} one defines[10] in the Minkowskian region invariant coupling function⁵

$$\alpha_{\text{an}}(Q^2; f) \rightarrow \tilde{\alpha}(s; f) = \mathbf{R}[\alpha_{\text{an}}] = \int_s^\infty \frac{d\sigma}{\sigma} \rho(\sigma; f) \quad (15)$$

or some other quantity like

$$R_\pi(s) \equiv \mathbf{R}[D_{\text{pt}}(Q^2)] = 1 + \sum_{k \geq 1} d_k \mathfrak{A}_k(s); \quad \mathfrak{A}_k(s) = \mathbf{R}[\bar{\alpha}_s^k(Q^2)] \quad (6)$$

⁵As it follows from this expression, the spectral function can be considered as a beta-function. However, contrary to Schwinger’s hope[37], this $\rho(s; f)$, being a spectral function for the Euclidean invariant coupling, happens to be the RG generator for another, Minkowskian, invariant coupling[15].

with

$$\mathfrak{A}_k(s) = \int_s^\infty \frac{d\sigma}{\sigma} \rho_k(\sigma) ; \quad \rho_k(\sigma) = \Im [\alpha_s^k(-\sigma)] . \quad (16)$$

Finally, we have a simple possibility of reconstructing an Euclidean object from the corresponding Minkowskian one with the help of the dipole operator \mathbf{D} like

$$\alpha_{\text{an}}(Q^2; f) = \mathbf{D} \{ \tilde{\alpha}(s; f) \} .$$

In particular, by substituting $\tilde{\alpha}^{(1)}(s; f)$ into the integrand, we obtain after integration by parts

$$\begin{aligned} \mathbf{D} \{ \tilde{\alpha}^{(1)} \} &= \frac{Q^2}{\pi\beta_0} \int_0^\infty \frac{d\sigma}{(\sigma + Q^2)^2} \cdot \left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{\ln(\sigma/\Lambda^2)}{\pi} \right) = \\ &= \frac{1}{\beta_0} \left[\frac{1}{\ln Q^2/\Lambda^2} + \frac{\Lambda^2}{\Lambda^2 - Q^2} \right] = \alpha_{\text{an}}^{(1)}(Q^2, f) . \end{aligned} \quad (17)$$

This simple calculation elucidates the relation between ghost-free expressions in the Minkowskian and Euclidean regions. They are related by a reverse transformation as well. For instance, in accordance with (15),

$$\tilde{\alpha}^{(1)}(s; f) = \mathbf{R} [\alpha_{\text{an}}^{(1)}(Q^2; f)] .$$

In Fig.1, we give a concise summary of the IAA results for invariant analytic couplings $\alpha_{\text{an}}(Q^2, 3)$ and $\tilde{\alpha}(s, 3)$ calculated for one-, two- and three-loop cases in both the Euclidean and Minkowskian domains.

Here, the dash-dotted curves represent the one-loop IAA approximations (2) and (17). The solid IAA curves are based on the exact two-loop solutions of RG equations⁶ and approximate three-loop solutions in the $\overline{\text{MS}}$ scheme. Their remarkable coincidence (within the 1-2 per cent limit) demonstrates reduced sensitivity of the IAA with respect to the higher-loops effects in the whole Euclidean and Minkowskian regions from IR to UV limits.

For comparison, by the dotted line we also give a usual $\bar{\alpha}_s(Q^2)$ two-loop effective QCD coupling with a pole at $Q^2 = \Lambda^2$.

As it has been shown in [21, 24, 25], relations parallel to eqs.(15) and (9) are valid for powers of the pQCD invariant coupling. This can be resummed in the form of a self-consistent scheme. Consider now new functional sets of nonpower perturbation expansions.

⁶As it has recently been established the exact solution to the two-loop RG differential equation for the invariant coupling can be expressed in terms of a special function W , the Lambert function, defined by the relation $W(z)e^{W(z)} = z$ with an infinite number of branches $W_n(z)$. For some details of analyticized and pipitized solutions expressed in terms of the Lambert function, see Refs. [31, 32, 33, 17, 34, 38].

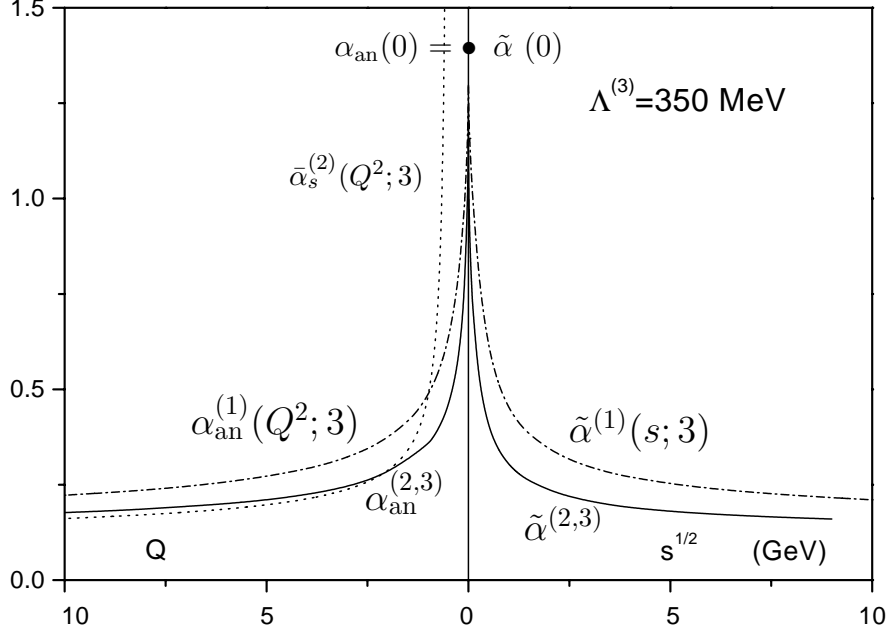


Figure 1: *Space-like and time-like invariant analytic couplings in a few GeV domain*

2.2 Expansion of observables over nonpower sets $\{\mathcal{A}\}$ and $\{\mathfrak{A}\}$

To realize the effect of transition from expansion over the “traditional” power set

$$\{\bar{\alpha}_s^k(Q^2, f)\} = \bar{\alpha}_s(Q^2), \bar{\alpha}_s^2, \dots \bar{\alpha}_s^k \dots$$

to expansions over nonpower sets in the space-like and time-like domains

$$\{\mathcal{A}_k(Q^2, f)\} = \alpha_{\text{an}}(Q^2, f), \mathcal{A}_2(Q^2, f), \mathcal{A}_3 \dots ; \{\mathfrak{A}_k(s, f)\} = \tilde{\alpha}(s, f), \mathfrak{A}_2(s, f), \mathfrak{A}_3 \dots ,$$

it is instructive to learn properties of the latter.

In a sense, both nonpower sets are similar

— They consist of functions that are free of unphysical singularities.

— First functions, the new effective couplings, $\mathcal{A}_1 = \alpha_{\text{an}}$ and $\mathfrak{A}_1 = \tilde{\alpha}$ are monotonically decreasing. In the IR limit, they are finite and equal $\alpha_{\text{an}}(0, 3) = \tilde{\alpha}(0, 3) \simeq 1.4$ with the same infinite derivatives. Both have the same leading term $\sim 1/\ln x$ in the UV limit.

— All other functions (“effective coupling powers”) of both the sets start from the zero IR values $\mathcal{A}_{k \geq 2}(0, f) = \mathfrak{A}_{k \geq 2}(0, f) = 0$ and obey the UV behavior $\sim 1/(\ln x)^k$ corresponding to $\bar{\alpha}_s^k(x)$. They are no longer monotonous. The second functions \mathcal{A}_2 and \mathfrak{A}_2 are positive with maximum around $s, Q^2 \sim \Lambda^2$. Higher functions $\mathcal{A}_{k \geq 3}$ and $\mathfrak{A}_{k \geq 3}$ oscillate in the region of low argument values and obey $k - 2$ zeroes.

Remarkably enough, the mechanism of liberation of unphysical singularities is quite different. While in the space-like domain it involves non-perturbative, power in Q^2 , structures, in the time-like region, it is based only upon resummation of the “ π^2 terms”. Figuratively, (non-perturbative !) *analyticization* in the Q^2 -channel can be treated as a quantitatively distorted reflection (under $Q^2 \rightarrow s = -Q^2$) of (perturbative) “*pipization*” in the s -channel. This effect of “distorting mirror” first discussed in [27] is illustrated in figures 1 and 2.

Summarize the main results essential for data analysis. Instead of power perturbative series in the space-like $D_{\text{pt}}(Q^2) = 1 + d_{\text{pt}}(Q^2)$

$$d_{\text{pt}}(Q^2) = \sum_{k \geq 1} d_k \bar{\alpha}_s^k(Q^2; f)$$

and time-like regions $R_{\text{pt}}(s) = 1 + r_{\text{pt}}(s)$

$$r_{\text{pt}}(s) = \sum_{k \geq 1} r_k \tilde{\alpha}^k(s; f); \quad (r_{1,2} = d_{1,2}, r_3 = d_3 - d_1 \frac{\pi^2 \beta_{[f]}^2}{3}, r_4 = d_4 - \dots),$$

one has to use asymptotic expansions (13) and (6)

$$d_{\text{an}}(Q^2) = \sum_{k \geq 1} d_k \mathcal{A}_k(Q^2, f); \quad r_{\pi}(s) = \sum_{k \geq 1} d_k \mathfrak{A}_k(s, f)$$

with *the same coefficients* d_k over non-power sets of functions $\{\mathcal{A}\}$ and $\{\mathfrak{A}\}$.

3 Global formulation of APT

To apply the modified APT to analyze QCD processes, it is necessary to formulate it “globally”, for the whole domain accessible to modern experiment, that is for regions with various flavour numbers f of active quarks. To this goal, one has to consider the issue of heavy quark threshold crossing.

3.1 Threshold matching.

In a real calculation, the procedure of the threshold matching is in use. One of the simplest is the matching condition in the massless $\overline{\text{MS}}$ scheme[39]

$$\bar{\alpha}_s(Q^2 = M_f^2; f - 1) = \bar{\alpha}_s(Q^2 = M_f^2; f) \quad (18)$$

related to the mass squared M_f^2 of the f -th quark.

This condition allows one to define a “global” function $\bar{\alpha}_s(Q^2)$ consisting of the smooth parts

$$\bar{\alpha}_s(Q^2) = \bar{\alpha}_s(Q^2; f) \quad \text{at} \quad M_{f-1}^2 \leq Q^2 \leq M_f^2 \quad (19)$$

and continuous in the whole space-like interval of positive Q^2 values with discontinuity of derivatives at the matching points. We call such functions the *spline-continuous* ones.

At first sight, any massless matching, yielding the spline-type function, violates the analyticity in the Q^2 variable, thus disturbing the relation between the s - and Q^2 -channels⁷.

However, in the IAA, the original power perturbation series (12) with its unphysical singularities and possible threshold non-analyticity has no direct relation to data, being a sort of a “raw material” for defining spectral density. Meanwhile, the discontinuous density is not dangerous. Indeed, an expression of the form

$$\rho_k(\sigma) = \rho_k(\sigma; 3) + \sum_{f \geq 4} \theta(\sigma - M_f^2) \{\rho_k(\sigma; f) - \rho_k(\sigma; f-1)\} \quad (20)$$

with $\rho_k(\sigma; f) = \Im \bar{\alpha}_s^k(-\sigma, f)$ defines, according to (14) and (16), the smooth global

$$\mathcal{A}_k(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma + x} \rho_k(\sigma) \quad (21)$$

and spline-continuous global

$$\mathfrak{A}_k(s) = \int_s^\infty \frac{d\sigma}{\sigma} \rho_k(\sigma) \quad (22)$$

functions⁸.

We see that in this construction the role of the input perturbative invariant coupling $\bar{\alpha}_s(Q^2)$ is twofold. It provides us not only with spectral density (20) but with matching conditions (18) relating Λ_f with Λ_{f+1} as well.

Note that the matching condition (18) is tightly related [39, 41] to the renormalization procedure. Just for this profound reason we keep it untouched (compare with Ref. [27]).

3.2 The s -channel: shift constants.

As a practical result, we now observe that the “global” s -channel coupling $\tilde{\alpha}(s)$ and other functions $\mathfrak{A}_k(s)$ generally differ from the effective coupling with a fixed flavor number f $\tilde{\alpha}(s; f)$ and $\mathfrak{A}_k(s; f)$ by constants. For example, at $M_5^2 \leq s \leq M_6^2$

$$\tilde{\alpha}(s) = \int_s^\infty \frac{d\sigma}{\sigma} \rho(\sigma) = \int_s^{M_6^2} \frac{d\sigma}{\sigma} \rho(\sigma; 5) + \int_{M_6^2}^\infty \frac{d\sigma}{\sigma} \rho(\sigma; 6) = \tilde{\alpha}(s; 5) + c(5).$$

⁷Any massless scheme is an approximation that can be controlled by the related mass-dependent scheme [40]. Using such a scheme, one can devise [41] a smooth transition across the heavy quark threshold. Nevertheless, from the practical point of view, it is sufficient (besides the case of data lying in close vicinity to the threshold) to use the spline-type matching (18) and forget about the smooth threshold crossing.

⁸Here, by eqs.(21),(22) and (20) we have introduced new “global” effective invariant couplings and higher expansion functions different from the previous ones with a fixed f value.

Generally,

$$\tilde{\alpha}(s) = \tilde{\alpha}(s; f) + c(f) \quad \text{at} \quad M_f^2 \leq s \leq M_{f+1}^2 \quad (23)$$

with *shift constants* $c(f)$ that can be calculated in terms of integrals over $\rho(\sigma; f+n)$ $n \geq 1$ with additional reservation $c(6) = 0$ related to the asymptotic freedom condition.

More specifically,

$$c(f-1) = \tilde{\alpha}(M_f^2; f) - \tilde{\alpha}(M_f^2; f-1) + c(f), \quad c(6) = 0.$$

These $c(f)$ reflect the $\tilde{\alpha}(s)$ continuity at the matching points M_f^2 .

Analogous shift constants

$$\mathfrak{A}_k(s) = \mathfrak{A}_k(s; f) + \mathfrak{c}_k(f) \quad \text{at} \quad M_f^2 \leq s \leq M_{f+1}^2 \quad (24)$$

are responsible for continuity of higher expansion functions. Meanwhile, $\mathfrak{c}_2(f)$ relates to discontinuities of the “main” spectral function (20).

The one-loop estimate with $\beta_{[f]}\rho(\sigma; f) = \{\ln^2(\sigma/\Lambda_f^2) + \pi^2\}^{-1}$,

$$c(f-1) - c(f) = \frac{1}{\pi\beta_{[f]}} \arctan \frac{\pi}{\ln \frac{M_f^2}{\Lambda_f^2}} - \frac{1}{\pi\beta_{[f-1]}} \arctan \frac{\pi}{\ln \frac{M_f^2}{\Lambda_{f-1}^2}} \simeq \frac{17-f}{54} \bar{\alpha}_s^3(M_f^2) \quad (25)$$

and traditional values of the scale parameter $\Lambda_3, \Lambda_4 \sim 350 - 250$ MeV reveal that these constants

$$c(5) \simeq 3.10^{-4}, \quad c(4) \simeq 3.10^{-3}, \quad c(3) \sim 0.01; \quad \mathfrak{c}_2(f) \simeq 3\alpha(M_f^2)c(f)$$

are essential at a few per cent level for $\tilde{\alpha}$ and at ca 10% level for \mathfrak{A}_2 .

This means that the quantitative analysis of some s -channel events like, e.g., e^+e^- annihilation [24], τ -lepton decay [25] and charmonium width [8] at the $f = 3$ region should be influenced by these constants.

3.3 Global Euclidean functions.

On the other hand, in the Euclidean, instead of the spline-type function $\bar{\alpha}_s$, we have now continuous, analytic in the whole $Q^2 > 0$ domain, invariant coupling defined, along with (21), via the spectral integral

$$\alpha_{\text{an}}(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma + Q^2} \rho(\sigma) \quad (26)$$

with the discontinuous density $\rho(\sigma)$ (20).

Unhappily, here, unlike the time-like region, there is no possibility of enjoying any more explicit expression for $\alpha_{\text{an}}(Q^2)$ even in the one-loop case. Moreover, the Euclidean functions α_{an} and \mathcal{A}_k , being considered in a particular f -flavour region $M_f^2 \leq Q^2 \leq M_{f+1}^2$, do depend on all $\Lambda_3, \dots, \Lambda_6$ values simultaneously.

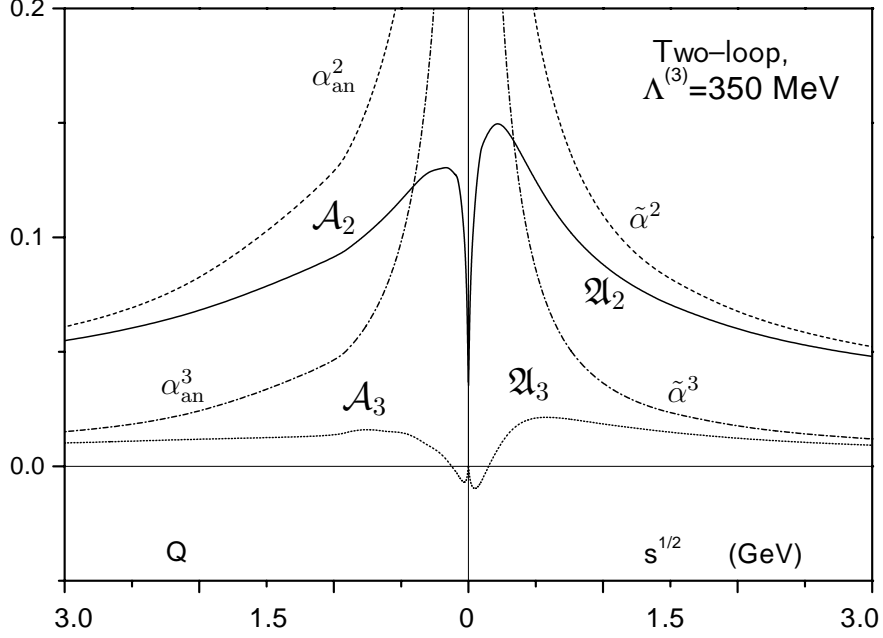


Figure 2: “Distorted mirror symmetry” for global expansion functions

Nevertheless, the real difference from the $f = 3$ case, numerically, is not big at small Q^2 and in the “few GeV region”, for practical reasons, it could be of importance .

This situation is illustrated by Fig. 2. Here, by thick solid curves with maxima around $\sqrt{s}, Q \equiv \Lambda$, we draw expansion functions \mathcal{A}_2 and \mathfrak{A}_2 in a few GeV region. Thin solid lines zeroes around Λ and negative values below, represent \mathcal{A}_3 and \mathfrak{A}_3 . For comparison, we give also second and third powers of relevant analytic couplings α_{an} and $\tilde{\alpha}$.

All these functions correspond to exact two-loop solutions expressed in terms of Lambert function ⁹.

4 Illustrations

Another quantitative effect stems from the nonpower structure of the IAA perturbative expansion. It is also emphasized at the few GeV region.

4.1 The s -channel

To illustrate the qualitative difference between our global scheme and common practice of data analysis, we first consider the $f = 3$ region.

⁹Details of these calculations can be found in Ref.[17]. Assistance of D.S. Kurashev and B.A. Magradze in calculation of curves with Lambert functions is gratefully acknowledged.

The process of Inclusive e^+e^- hadron annihilation provides us with an important piece of information on the QCD parameters. In the usual treatment, (see, e.g., Refs.[4, 6]) the basic relation can be presented in the form

$$\frac{R(s)}{R_0} = 1 + r(s); \quad r_{pt}(s) = \frac{\bar{\alpha}_s(s)}{\pi} + r_2 \bar{\alpha}_s^2(s) + r_3 \bar{\alpha}_s^3(s). \quad (27)$$

Here, the numerical coefficients $r_1 = 1/\pi = 0.318$, $r_2 = 0.142$, $r_3 = -0.413$ (given for the $f = 5$ case) are not diminishing. However, a rather big negative r_3 value comes mainly from the $-r_1\pi^2\beta_{[5]}^2/3$ contribution equal to -0.456 . Instead of (27), with due account of (6), we now have

$$r_\pi(s) = \frac{\tilde{\alpha}(s)}{\pi} + d_2 \mathfrak{A}_2(s) + d_3 \mathfrak{A}_3(s); \quad (28)$$

with reasonably decreasing coefficients $d_1 = 0.318$; $d_2 = 0.142$; $d_3 = 0.043$, the mentioned π^2 term of r_3 being “swallowed” by $\tilde{\alpha}(s)$ ¹⁰

Now, the main difference between (28) and (27) is due to the term $d_2 \mathfrak{A}_2$ standing in the place of $d_2 \tilde{\alpha}^2$. The difference can be estimated by adding into (27) the structure $r_4 \alpha^4$ with $r_4 \simeq -1$. This effect could be essential in the region of $\tilde{\alpha}(s) \simeq 0.20 - 0.25$. Here, in the APT analysis, the third, three-loop term contributes about half of a per cent, compared with 5,5% in the usual case.

The APT algorithm with fixed $f = 3$ has recently been used [43] for the analysis of Inclusive τ -decay. Here, the theoretical expression for an observed quantity, the time-life of τ lepton, contains QCD correction Δ expressed via an integral of an s -channel matrix element over the region $0 < s < M_\tau^2$.

As a result of the three-loop analysis of a modern [44] experimental value $\Delta_{exp}(s_0 = 3.16^2) = 0.191$, it was obtained that $\tilde{\alpha}(M_\tau^2) = 0.380$. Remind here that under usual treatment one obtains $\bar{\alpha}_s(M_\tau^2) = 0.334$ that can hardly be related to any $\bar{\alpha}_s(M_\tau^2)$ value as far as the parameter $\pi^2/\ln^2(M_\tau^2/\Lambda^2)$ is close to unity.

Note also that the third term of (28) contributes here about one per cent.

4.2 The Q^2 -channel : Sum Rules

In the Euclidean channel, instead of power expansion like (12), we typically have

$$d(Q^2) = \frac{\alpha_{an}(Q^2)}{\pi} + d_2 \mathcal{A}_2(Q^2) + d_3 \mathcal{A}_3(Q^2). \quad (29)$$

Here, the modification is related to a non-perturbative power structures behaving like Λ^2/Q^2 at $Q^2 \gg \Lambda^2$. As it has been estimated above, these corrections could be essential in a few GeV region.

¹⁰This term contributes about 8.10^{-4} to the $r(M_Z^2)$ and, correspondingly, 0.0025 to the extracted $\bar{\alpha}_s(M_Z^2)$ value. This means that the main part of the “traditional three-loop term” $r_3 \bar{\alpha}_s^3$ in the r.h.s. of (27), being of the one-loop origin, is essential for the modern quantitative analysis of the data. In particular, it should be taken into the account even in the so-called NLLA which is a common approximation for the analysis of events at $\sqrt{s} \sim M_Z$. For a more detailed numerical APT analysis of the $f = 5$ region, see [42].

In the paper [26], the IAA has been applied to the Bjorken sum rules. Here, one has to deal with the Q^2 -channel at small transfer momentum squared $Q^2 \lesssim 10 \text{ GeV}^2$.

Due to some controversy of experimental data, we give here only a part of the results of [26]. For instance, using data of the SMC Collaboration [45] for $Q_0^2 = 10 \text{ GeV}^2$, the authors obtained $\alpha_{\text{an}}^{(3)}(Q_0^2) = 0.301$ instead of $\alpha_{\text{pt}}^{(3)}(Q_0^2) = 0.275$. Here, the contribution of the third term is also suppressed.

The same remark is valid in the analysis of the Gross–Llywellin-Smith (GLS) sum rules. Indeed, as it was shown in paper [28], instead of proportions $(65 : 24 : 11)_{\text{TB}}$ ¹¹ of usual analysis, the APT gives $(75 : 21 : 4)_{\text{APT}}$ (for further details, see Section IIc in [28]). The same effect for the Bjorken sum rules turns out[26] to be more pronounced $(55 : 26 : 19)_{\text{TB}} \rightarrow (80 : 19 : 1)_{\text{ATB}}$.

Some comments are in order:

— We see that, generally, the extracted values of α_{an} and of $\tilde{\alpha}$ are both slightly greater in a few GeV region than the relevant values of $\bar{\alpha}_s$ for the same experimental input. This corresponds to the above-mentioned non-power character of new asymptotic expansions with a suppressed higher-loop contribution.

— At the same time, for equal values of $\alpha_{\text{an}}(x_*) = \tilde{\alpha}(x_*) = \bar{\alpha}_s(x_*)$, the analytic scale parameter Λ_{an} values extracted from α_{an} and $\tilde{\alpha}$ are a bit greater than that $\Lambda_{\overline{\text{MS}}}$ taken from $\bar{\alpha}_s$. This feature is related to a “smoother” behavior of both the regular functions α_{an} and $\tilde{\alpha}$, as compared to the singular $\bar{\alpha}_s$.

4.3 Conclusion

To summarize, we repeat once more our main points.

1. We have formulated a self-consistent scheme for analyzing data in both the space-like and time-like regions.

The fundamental equation connecting these regions is the dipole spectral relation (1) between renormalization-group invariant nonpower expansions $D_{\text{an}}(Q^2)$ and $R_{\pi}(s)$.

Just this equation (equivalent to the Källén–Lehmann representation), treated as a transformation, is responsible for non-perturbative terms in the Q^2 -channel involved into $\alpha_{\text{an}}(Q^2)$ and non-power expansion functions $\{\mathcal{A}_k(Q^2)\}$. These terms, non-analytic in the coupling constant α , are a counterpart to the perfectly perturbative π^2 -terms effectively summed in the s -channel expressions $\tilde{\alpha}(s)$ and $\{\mathfrak{A}_k(s)\}$.

2. As a by-product, we ascertain a new qualitative feature of the IAA, relating to its non-perturbativity in the Q^2 -domain. It can be considered as a **minimal non-perturbativity** or **minimal non-analyticity**¹² in α as far as it corresponds to perturbativity in the s -channel.

Physically, it implies that minimal non-perturbativity cannot be referred to any mechanism producing effect in the s -channel.

¹¹That is, the contribution of the first, linear in $\bar{\alpha}_s$, is 65 % , while the contributions of the second and third are 24 and 11 per cent.

¹²Compatible with the RG invariance and the Q^2 analyticity — compare with [46].

3. The next result relates to the correlation between regions with different values of the effective flavor number f . Dealing with the massless $\overline{\text{MS}}$ renormalization scheme, we argue that the usual perturbative QCD expansion provides our scheme only with step-discontinuous spectral density (20) depending simultaneously on different scale parameters Λ_f ; $f = 3, \dots, 6$ connected by usual matching relations.

This step-discontinuous spectral density yields, on the one hand, smooth analytic coupling $\alpha_{\text{an}}(Q^2)$ and higher functions $\{\mathcal{A}_k(Q^2)\}$ in the space-like region — eq.(21).

On the other hand, it produces the spline-continuous invariant coupling $\tilde{\alpha}(s)$ and functions $\{\mathfrak{A}_k(s)\}$ in the time-like region — eq.(22).

As a result, the global expansion functions $\{\mathcal{A}_k(Q^2)\}$ and $\{\mathfrak{A}_k(s)\}$ differ both from the ones $\{\mathcal{A}_k(Q^2; f)\}$ and $\{\mathfrak{A}_k(s; f)\}$ with a fixed value of a flavour number.

4. Thus, our global APT scheme uses the common invariant coupling $\bar{\alpha}_s(Q^2, f)$ and matching relations, only as an input. Practical calculation for an observable now involves expansions over the sets $\{\mathcal{A}_k(Q^2)\}$ and $\{\mathfrak{A}_k(s)\}$, that is non-power series with usual numerical coefficients d_k obtained by calculation of the relevant Feynman diagrams.

In particular, this means that we have now **three QCD effective couplings**: $\tilde{\alpha}$, α_{an} — of the APT formalism, and traditional $\bar{\alpha}_s$. This usual one can be used for approximate expression of two first ones in four and five-flavor regions, for the comparison reasons.

This means that, generally, one should check the accuracy of the bulk of extractions of the QCD parameters from diverse “low energy” experiments. Our preliminary estimate shows that such a revision could influence the rate of their correlation.

5. Last but not least. As it has been mentioned in our recent publications [21, 24], the IAA obeys immunity with respect to higher loop and renormalization scheme effects.

Now, we have got an additional insight into this item related to observables and can state that the perturbation series for an observable in the IAA have better convergence properties (than in the usual RG-summed perturbation theory) in both the s - and Q^2 -channels.

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